Modal Decomposition Applied to Heat Conduction

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1. Introduction

This paper presents the state-variable modal decomposition of transient temperature ensemble data of into modal components to estimate time constants and characteristic shapes of a heated rod. The equation for heat conduction in a one-dimensional continuum is

$$\frac{\partial y}{\partial t} = \alpha \frac{\partial^2 y}{\partial x^2}, \tag{1}$$

where $y(x,t)$ is temperature, t is time, x is the spatial coordinate, and α is a constant (thermal diffusivity), with initial conditions and boundary conditions. Solution by separation of variables, such that $y(x,t) = Y(x)q(t)$, leads to a differential eigenvalue problem, with eigenfunctions $Y_i(x)$ that go with $q_i(t) = e^{-\lambda_i t}$, where the λ_i are associated with eigenvalues.

If discretized, for example by finite elements, equation (1) is approximated by a system of first-order ordinary differential equations $\mathbf{A}\dot{\mathbf{y}} = \mathbf{B}\mathbf{y}$, where $\mathbf{y}$ is an M-dimensional vector of temperatures associated with discrete points on the continuum, and $\mathbf{A}$ and $\mathbf{B}$ are $M \times M$ matrices. The elements of $\mathbf{y}$ are state variables. Seeking solutions $\mathbf{y}(t) = \exp(-\lambda t)\mathbf{u}$, and inserting into $\mathbf{A}\dot{\mathbf{y}} = \mathbf{B}\mathbf{y}$, leads to the eigenvalue problem $-\hat{\lambda}\mathbf{A}\mathbf{u} = \mathbf{B}\mathbf{u}$, or, in matrix form

$$-\mathbf{A}\mathbf{U}\underline{\Lambda} = \mathbf{B}\mathbf{U}, \tag{2}$$

where $\mathbf{U}$'s columns are eigenvectors $\mathbf{u}_i$, and diagonal $\underline{\Lambda}$ has values $\hat{\lambda}_i$, for $i = 1, \ldots, M$. If the discretization is faithful, the vectors $\mathbf{u}_i$ and values $\hat{\lambda}_i$ approximate discretizations of eigenfunctions $Y_i(x)$ and continuous-system exponents λ_i, for the lower values of i.

If the $\mathbf{y}(t)$ values are sampled from an experiment, at times $t = 0, \Delta t, 2\Delta t, \ldots, (N-1)\Delta_t$, then an $M \times N$ measurement ensemble matrix $\mathbf{Y}$ can be formed such that the rows are time histories of the measured states. The ensemble of time derivatives can be estimated as $\mathbf{V} = \mathbf{V}\mathbf{D}^T$, using an $(N-2h) \times N$ matrix $\mathbf{D}$ of centered finite differences of span h. Thus $\mathbf{V}$ is $n \times (N-2h)$. The first and last h columns of $\mathbf{Y}$ are dropped. Then we form a correlation matrix $\mathbf{R} = \mathbf{Y}\mathbf{Y}^T/(N-2h)$ and a nonsymmetric matrix $\mathbf{N} = \mathbf{Y}\mathbf{V}^T/(N-2h)$. The state-variable modal decomposition eigenvalue problem is

$$\mathbf{R}\underline{\Psi}\underline{\Lambda} = \mathbf{N}\underline{\Psi}. \tag{3}$$

For multi-modal free responses of linear systems, the eigenvalues from equation (3) approximate the eigenvalues equation (2), and $\mathbf{U} = \underline{\Psi}^{-T}$ [1]. This decomposition method has been applied to vibration systems, for which eigenvalues are generally complex, and contain information about frequency and decay rate [1].

In this paper, we present the application of this method to the simulated heat conduction in a bar to extract its characteristic mode shapes and exponential rates.

2. Example

Consider the 1-D transient problem denoted by X12B10T0 (notation of Beck *et al.* [2]) of a bar with a suddenly applied temperature T_0 at one end, and insulated at the other. The governing equation (1) has the boundary conditions $y(0,t) = T_0$ and $[\partial y(x,t)/\partial x]_{x=L} = 0$, with the initial conditions $y(x,0) = 0$. The solution by separation of variables is

$$y(x,t) = T_0 - 2T_0 \sum_{m=0}^{\infty} \frac{\sin \beta_m x}{\beta_m L} e^{-\beta_m^2 \alpha t}, \tag{4}$$

where $\beta_m = (m - 1/2)\pi/L$, for $m = 1, 2, \ldots$, and $\beta_m^2 \alpha = \lambda_m$.

For the numerical example, we used $L = 1$ m, $\alpha = 1$ m^2/s, and $T_0 = 1$ degree. We sensed the rod at $M = 16$ locations $x = 0, \Delta x, 2\Delta x, \ldots, L$, where $\Delta x = L/(M-1)$. We set the time sampling interval Δt to be one tenth (arbitrary criterion) of the fastest desired time constant, $\tau_m = 1/\lambda_m$. We aimed for roughly five time constants, and thus set $\Delta t = \tau_5/10 = 0.0005$ s (or 2 kHz). We also set the total number of samples $N = 809$ (for a time of 0.4053 seconds) to be one fourth (arbitrary choice) of the settling time of the slowest time constant. We evaluated the separation of variables solution (4) truncated at $m = 16$ terms to generate an ensemble matrix $\mathbf{Y}$ with elements $Y_{ij} = y(x_i, t_j)$, for $i = 1, \ldots, M$ and $j = 1, \ldots, N$.

The decomposition eigenvalue problem was constructed as described above, with difference step $h = 2$. The true values of λ_i, $i = 1, \ldots, 9$, were 2.4674, 22.2066, 61.6850, 120.9027, 199.8595, 298.5555, 416.9908, 555.1652, and 713.0789 s^{-1}. The extracted values were 2.4674, 22.2084, 61.7241, 121.1994, 201.3453, 300.8649, 438.2915, 585.2685, and 707.0017 s^{-1}. Further extractions were not accurate. The first four estimated mode shapes, shown in Figure 1, resemble the theoretical sinusoids. Modal coordinates, or separated time-dependent variables, can be obtained from the extracted modes, as $\mathbf{Q} = \mathbf{U}^{-1}\mathbf{Y}$. The lowest modal coordinates are plotted in Figure 2, demonstrating the characteristic exponential decays. When random noise was applied, uniformly distributed between $\pm 2^{-9}$, two mode shapes were reasonably estimated. There was also an extracted mode of $\lambda_0 = 0$, with a vector discretization of $Y_0(x) = 1$, and $q_0(t) = 1$ representing the eventual steady state.

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4. References

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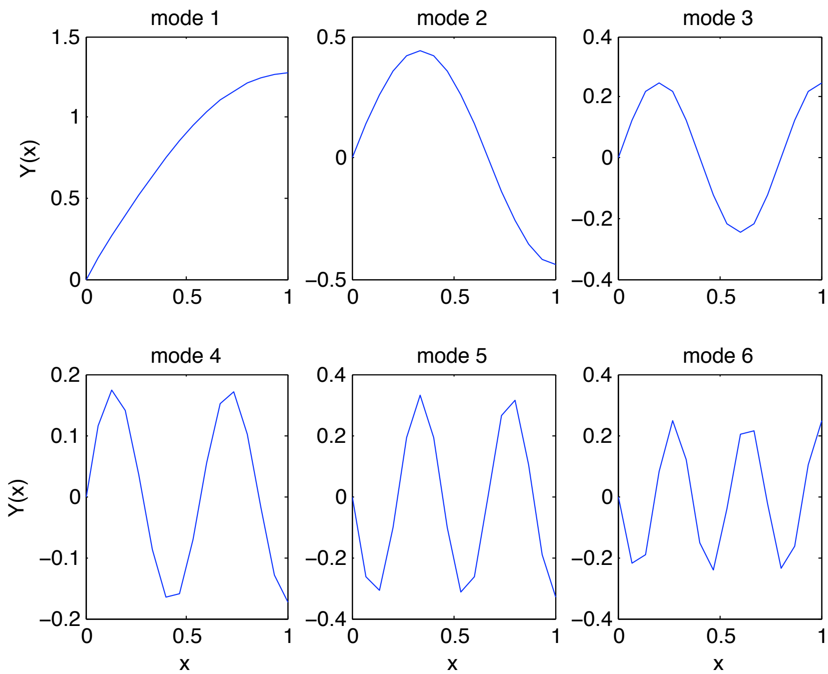

Figure 1: Extracted modal vectors approximate the modal functions $Y_i(x)$.

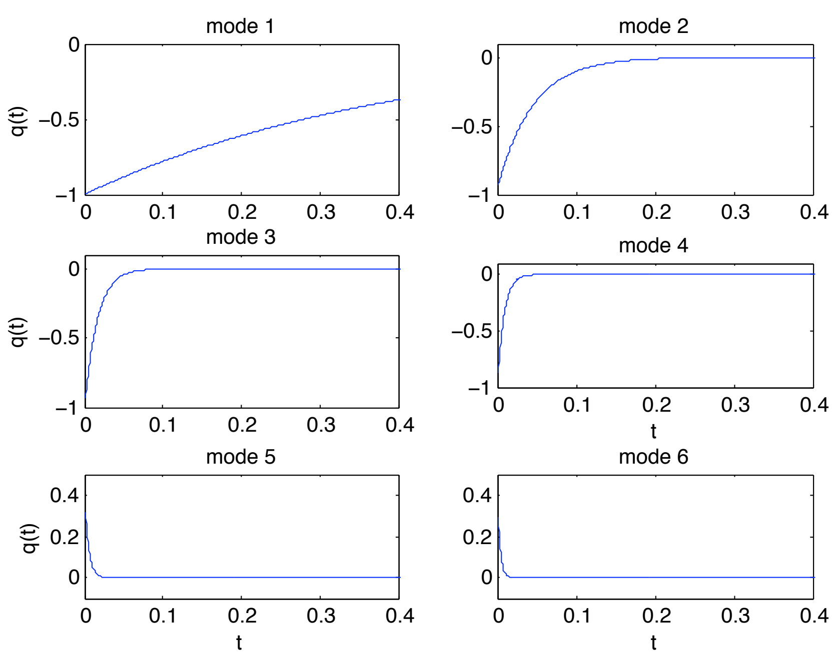

Figure 2: Transient responses of six separated variables as extracted from the data.